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First Passage Times for Correlated Random Walks and Some Generalizations

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It is generally difficult to solve Fokker–Planck equations in the presence of absorbing boundaries when both spatial and momentum coordinates appear in the boundary conditions. In this note we analyze a simple, exactly solvable model of the correlated random walk and its continuum analogue. It is shown that one can solve for the moments recursively in one dimension in exact analogy with first passage problems for the Fokker–Planck equation, although the boundary conditions are somewhat more complicated. Further generalizations are suggested to multistate random walks.

KEY WORDS: Random walks; first passage times; telegraphers equation; correlated random walks; multistate random walks.

1. INTRODUCTION

It has long been known that the boundary condition for a system described by a Fokker–Planck (FP) equation and an absorbing boundary S is $p = O$ for all $\mathbf{r} \in S$, provided that the FP equation contains spatial but not momentum coordinates. When momentum or velocity coordinates appear in the FP equation the formulation and solution of the resulting problem becomes more complicated, as was first pointed out by Wang and Uhlenbeck.[(1)] There are several studies of systems with absorbing boundaries in which velocity conditions must be obeyed for the impinging particles to be absorbed at a surface.[(2–5)] However, there is one simple case which does not seem to have been analyzed earlier and which can be solved exactly. In this note we present results for first passage times of the continuum limit of the simplest correlated random walk[(6–8)] and some of its generalizations. It has been remarked that the correlated random walk as discussed here, is the

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simplest example of a multistate random walk.[10] The correlated random walk is a multistate random walk because the state probabilities depend both on the site and direction of motion. Thus, the present work can be regarded as the simplest example of the first passage time problem for such random walks. For an extensive mathematical discussion of the solutions of the telegrapher's equation in unbounded space the reader is referred to the paper by Goldstein.[8] There are several papers on multistate random walks in unbounded space.[11–15]

2. FIRST PASSAGE TIME STATISTICS FOR THE CORRELATED RANDOM WALK

The simplest case to be considered is a one-dimensional correlated random walk confined to an interval $(-L, L)$ in which both end points are absorbing provided that the random walk is in the appropriate state. The distinct states correspond to momentum in a physical description. In the present model this translates into the requirement that the random walker be moving in the right direction when it impinges on an absorbing point. We need not restrict ourselves to the case in which the switching probabilities are constant in space, but will consider the generalization to both the discrete and continuous telegraphers equations with space-dependent switching probabilities. Let us therefore define

$$\begin{aligned} a_n(r) &= \Pr\{\text{First passage time} = n \text{ starting from site } r \,|\\ &\qquad \text{first step goes from } r \text{ to } r+1\} \\ b_n(r) &= \Pr\{\text{First passage time} = n \text{ starting from site } r \,|\\ &\qquad \text{first step goes from } r \text{ to } r-1\} \\ \alpha(r) &= \Pr\{\text{Random walk does not switch directions ar } r\} \\ \beta(r) &= 1 - \alpha(r) \end{aligned}$$

These definitions imply the set of equations

$$\begin{aligned} a_{n+1}(r) &= \alpha(r+1)\, a_n(r+1) + \beta(r+1)\, b_n(r+1) \\ b_{n+1}(r) &= \beta(r-1)\, a_n(r-1) + \alpha(r-1)\, b_n(r-1) \end{aligned} \tag{1}$$

together with the boundary conditions $a_n(L) = b_n(-L) = 0$. The continuum limit is obtained by scaling space and time by $t = n\Delta t$, $x = r\Delta L$ where Δt and $\Delta L \to 0$ in such a way that

$$c = \lim_{\Delta t, \Delta L \to 0} \Delta L/\Delta t \tag{2}$$

The switching probability $\alpha(r)$ in the limiting process will be assumed to take the form

$$\alpha(r) = 1 - \Delta t/(2T(r)), \qquad \beta(r) = \Delta t/(2T(r)) \tag{3}$$

where $T(r)$ has the dimensions of time. On taking the limits Δt, $\Delta L \to 0$ and setting $x = r\Delta L$, we find that $a(x, t)$ and $b(x, t)$ satisfy the coupled set of equations

$$\frac{\partial a}{\partial t} = c\frac{\partial a}{\partial x} + \frac{1}{2T(x)}(b - a) \tag{4a}$$

$$\frac{\partial b}{\partial t} = -c\frac{\partial b}{\partial x} + \frac{1}{2T(x)}(a - b) \tag{4b}$$

which is to be solved subject to the boundary conditions $a(L, t) = b(-L, t) = 0$. It is both convenient and makes some physical sense to consider, instead of $a(x, t)$ and $b(x, t)$ the variables

$$\begin{aligned} u &= a + b \\ v &= a - b \end{aligned} \tag{5}$$

We find that u and v satisfy

$$\begin{aligned} \frac{\partial u}{\partial t} &= c\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial t} &= c\frac{\partial u}{\partial x} - \frac{v}{T(x)} \end{aligned} \tag{6}$$

or, on eliminating u,

$$\begin{gathered} \frac{\partial^2 v}{\partial t^2} + \frac{1}{T(x)}\frac{\partial v}{\partial t} = c^2\frac{\partial^2 v}{\partial x^2} \\ \frac{\partial u}{\partial t} = c\frac{\partial v}{\partial x} \end{gathered} \tag{7}$$

Hence v satisfies a spatially dependent telegrapher's equation and once v is known u can be found by integrating with respect to t. However, these equations are deceptively easy to solve since the boundary conditions are coupled. If instead of using the functions u and v one remains with equations in terms of a and b then the boundary conditions on $a(x, t)$ alone are

$$a(L, t) = 0, \qquad a(-L, t) + 2T(-L)\left(\frac{\partial a}{\partial t} - c\frac{\partial a}{\partial x}\right)\Bigg|_{x=-L} = 0 \tag{8}$$

The secondary boundary condition is found by solving for b in terms of a from Eq. (4a).

3. MOMENTS OF FIRST PASSAGE TIME

While it would clearly pose a formidable problem to solve Eqs. (4) or (6) for even the most trivial choice $T(x) = \text{const}$, it is possible to use Eq. (4) to calculate moments of the first passage time in one dimension, analogous to the situation for one dimensional Fokker–Planck equations.[16,17] Denote the nth moments of the first passage times starting from x and initially moving to the right (left) to be $\mu_n(x)(\upsilon_n(x))$. These are defined by the relations

$$\mu_n(x) = \int_0^\infty t^n a(x, t)\, dt, \qquad \upsilon_n(x) = \int_0^\infty t^n b(x, t)\, dt \tag{9}$$

If we multiply Eq. (4) by t^n and integrate with respect to t we find

$$\begin{aligned} -n\mu_{n-1}(x) &= c\,\frac{d\mu_n(x)}{dx} + \frac{1}{2T(x)}\,[\upsilon_n(x) - \mu_n(x)] \\ -n\upsilon_{n-1}(x) &= -c\,\frac{d\upsilon_n(x)}{dx} + \frac{1}{2T(x)}\,[\mu_n(x) - \upsilon_n(x)] \end{aligned} \tag{10}$$

where $\mu_0(x) = \upsilon_0(x) = 1$. Consider the calculation of the first passage times, $n = 1$. For this it is convenient to introduce the auxiliary functions

$$U = \mu_1 + \upsilon_1, \qquad V = \mu_1 - \upsilon_1 \tag{11}$$

corresponding to u and v which satisfy

$$\begin{aligned} U' &= 2\rho(x)\, V \\ V' &= -\frac{2}{c} \end{aligned} \tag{12}$$

where $\rho(x) = [2cT(x)]^{-1}$. This set of equations is easily solved since one can find both U and V by successive integrations. One finds in this way that

$$\begin{aligned} U(x) &= 2A \int_{-L}^{x} \rho(\xi)\, d\xi + B - \frac{4}{c} \int_{-L}^{x} \xi\rho(\xi)\, d\xi \\ V(x) &= A - \frac{2x}{c} \end{aligned} \tag{13}$$

where A and B are the constants

$$\begin{aligned} A &= \frac{2}{c} \int_{-L}^{L} \xi\rho(\xi)\, d\xi \Big/ \left(1 + \int_{-L}^{L} \rho(\xi)\, d\xi\right) \\ B &= A + 2L/c \end{aligned} \tag{14}$$

Notice that when $T(x) = T(-x)$ so that there is symmetry around the origin, $A = 0$ from which it follows that $U(x) = U(-x)$. Since it would ordinarily not be possible to distinguish $\mu_1(x)$ and $\upsilon_1(x)$, the function $U(x)$ plays the major role. In particular, if $\rho(x) = 1/L_0 = \text{const}$ then $U(x)$ is

$$U(x) = \frac{2L}{c} - \frac{4}{cL_0}(x^2 - L^2) \tag{15}$$

which clearly differs from zero at $x = \pm L$. This is to be expected since the possibility exists that the diffusing particle is moving in the wrong direction when it reaches an end point. Higher moments of the first passage time can be found by using exactly the same argument as for the first moment, as can the probability of being absorbed at a particular point.

4. GENERALIZED MULTISTATE WALKS

Although there is historical precedent for studying the particular correlated random walk whose diffusive properties are described by the telegrapher's equation is should be remarked that this is only one example of the more general class of multistate random walks. Goldstein[8] has mentioned briefly some generalizations that he had considered that led to partial differential equations of order greater than 2, without presenting any details. Such generalizations can be produced in many different ways. For example, consider again a one dimensional random walk in which the random walker can exist in any one of $k > 2$ states. Let $a_{i,n}(r)$ be the joint probability that the random walker is in state i, at position r, and that the first passage time to absorption is equal to n. Let us suppose that a random walker in state i will make a step to the right with probability θ_i or a step to the left with probability $1 - \theta_i$. Let us further assume that at each step the probability that the state of a random walker at site r changes from i to j is $\alpha_{ij}(r)$. Then Eq. (1) is replaced by

$$\begin{aligned} a_{i,n+1}(r) = \theta_i \sum_j \beta_{ij}(r+1)\, a_{j,n}(r+1) \\ + (1-\theta_i) \sum_j \beta_{ij}(r-1)\, a_{j,n}(r-1) \end{aligned} \tag{16}$$

If one adds the scaling requirements

$$\begin{aligned} &\beta_{ii}(r) = 1 - \Delta t/T_i(r) \\ &\beta_{ij}(r) = \Delta t/T_{ij}(r), \qquad i \neq j \\ &x = r\Delta L, \qquad r = 0, 1, 2,..., \qquad c = \lim_{\Delta L, \Delta t \to 0} \Delta L/\Delta t \end{aligned} \tag{17}$$

where

$$\frac{1}{T_i(r)} = \sum_{j(\neq i)} \frac{1}{T_{ij}(r)} \tag{18}$$

then in the limit $\Delta L, \Delta t \to 0$ Eq. (16) goes over into the set of equations

$$\frac{\partial a_i}{\partial t} = (2\theta_i - 1)c\frac{\partial a_i}{\partial x} + \sum_j (a_j - a_i)\frac{1}{T_{ij}(x)} \tag{19}$$

generalizing Eq. (4). These reduce to the two state problem corresponding to the correlated random walk if we choose $k = 2$, $\theta_1 = 1$, $\theta_2 = 0$, $T_{12} = T_{21} = T$. As before one can write equations for first passage time moments analogous to Eq. (10) except that it would be difficult to solve them for general $T_{ij}(x)$ as we have in the $k = 2$ case.

Other generalizations of the correlated random walk are possible which include longer range steps or analogues of the continuous time random walk.[18] It is interesting to note that since the random walks discussed so far are basically Markoffian the distribution for displacements in the absence of boundaries must tend to the Gaussian form by the central limit theorem. This was proved in detail by Goldstein[8] for the simplest correlated random walk. It is not known whether it follows that the associated first passage time densities must approach the density associated with a Gaussian, or the conditions that must be satisfied to insure that it does.

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